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A First-Order Level-2 Phase Transition in Thermodynamic Formalism

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We show the existence of a phase transition at the level of measures for the generalized dimension of the maximal entropy measure in a model that was considered by F. Hofbauer and which is related to a model of M. Fisher. The model presented here is related to the one-dimensional Ising model in which a wall effect is assumed. In this situation, the problem has to be considered in the one-dimensional lattice $\mathbb{N}$. In general there is no first-order transition for the Ising model in the lattice $\mathbb{Z}$, but under our assumptions such transitions can occur. The Ising model has the purpose of explaining the magnetization of ferromagnetic systems at low temperatures. The main difference of our result from a previous result of F. Hofbauer is that the transition is analyzed in the setting of the generalized dimension. This setting is more closely related to the observables. The main purpose of this paper is to explain another mathematical model for phase transition using the mathematical results obtained by F. Hofbauer. We also use results of the thermodynamic formalism in an essential way.

KEY WORDS: Phase transition; thermodynamic formalism; entropy; pressure; generalized dimension.

INTRODUCTION

The generalized dimension was introduced some years ago and has been considered and analyzed by several authors. In ref. 13, we introduced some other mathematical elements in the definition of generalized dimension and in this way have a completely rigorous well-defined model that can be analyzed in several different situations. We also showed,[13] following the

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definition and the main features of the model, the existence of phase transitions.

The important features of the new elements introduced in the definition of generalized dimension are related to the pressure, equilibrium states associated with the pressure, and the concept (previously introduced by L. S. Young) of Hausdorff dimension of a measure.

It is well known that the values of the capacity dimension (or box dimension) and Hausdorff dimension of a general set are not always the same.

From a result of Young[(24)] (using a concept that roughly speaking concerns the capacity dimension of a measure and was previously introduced by F. Ledrapier), when considering the Hausdorff dimension of a measure and the capacity dimension of a measure, the two concepts will agree in general situations (see considerations at the end of the paper).

In our case this is an essential element in the understanding of the problem of phase transition and generalized dimension.

Concepts related to the capacity dimension are more natural for analyzing the phase transition than the concept of Hausdorff dimension[(6,7,19,21,22)] because it deals with the observables.

The situation that we will present here is not exactly the same as in ref. 13, but the same ideas can be applied in both cases.

After ref. 13 was finished, we noticed that the phenomenon of a jump from an equilibrium state to another in a discontinuous fashion was also present in a model of Fisher[(3,4)] as presented by Hofbauer.[(8)] This discontinuity happens in the setting of pressure and we will be able to analyze here the problem in the setting of generalized dimension.[(7,22)]

We believe we can present here the ideas that relate pressure and generalized dimension in a more transparent way, due to the fact that our map is simpler to deal with.

We will have to present a slightly different definition of generalized dimension that is suitable for the problem. This definition gives the correct model for understanding generalized dimension in the problem.

The discontinuity of the derivative of the pressure is transferred to the same property for the generalized dimension by means of a Legendre transform. The pressure that physicists consider (most of the time) is less than the pressure of thermodynamic formalism considered here.

We refer the reader to the paper of Hofbauer[(8)] for a more detailed mathematical framework and to the papers of Fisher[(4)] for the physical problems where the model was derived.

We also sugest that the reader who wishes more mathematical details read first refs. 8, 12, 13, 17, 20, and 23 and the final remarks of Section 2 before beginning to read Section 1.

Here we will say that we have a first-order (respectively second-order) transition in the value t_0 if the generalized dimension has lack of differentiability (respectively lack of continuity of the derivative of the generalized dimension) in the valule t_0. We will say (as in ref. 13) that we have a level-2 phase transition in the model if the probability laws one has to follow in choosing at random the centers of the balls for the coverings used in the definition of generalized dimensions (see Section 2 for complete definitions) have a discontinuous jump from one probability to a different one (see Section 2).

In a forthcoming paper we will consider the Felderhof–Fisher critical exponent of a transition.[17]

We will show here the following theorem:

Theorem. The Fisher model presents a first-order level-2 phase transition for the generalized dimension.

The proof of the theorem will be given in Section 2.

The results presented here can be extended to hyperbolic rational maps and more general Jacobians. We prefer here to state more modest results that can be more easily understood.

In Section 1, we make some general comments about the phase transition.

In Section 3 we present an example where one can compute explicitly the results presented in Sections 1 and 2.

1. GENERAL CONSIDERATIONS ABOUT PHASE TRANSITIONS

Consider a one-dimensional semilattice where each position is designated by a natural number $n = 0, 1, 2, 3, 4,...$. Suppose in each position n we have the possibility of two spins, positively or negatively oriented. We will associate these two possibilities respectively with 0 and 1.

In statistical mechanics, we are interested in the probability of a certain possible arrangement of spins in the lattice. Therefore, we are looking for a probability μ in $\{0, 1\}^{\mathbb{N}}$ that asserts for each possible subset (Borelean set) A of $\{0, 1\}^{\mathbb{N}}$ a number $\mu(A)$ giving the probability of such an event.

Suppose now we have an external parameter t (temperature, for example), such that for each value of t, we have μ_t the equilibrium probability law that gives the probability of each possible arrangement of spins as above.

A phase transition occurs, for instance, when there exists a certain transition value of the parameter t_0 such that the family μ_t has a sudden discontinuity at t_0.

This can happen, for example, for magnetic systems due to a sudden magnetization of the lattice. One possibility, for instance, could happen when in any possible position $n \in \mathbb{N}$ in the lattice, we have a spin negatively oriented (the arrangement $\{1, 1, 1, 1,...\}$).

This could be described in terms of measures by saying that the probability after the transition parameter t_0 is equal to a delta-Dirac with mass one in the arrangement $\{1, 1, 1,...\}$.

We will analyze this occurrence in a different but equivalent situation in Section 2. The phase space will not be $\{0, 1\}^{\mathbb{N}}$, but the interval $[0, 1]$. We will consider the problem in the context of iterations of a one-dimensional map on the interval $[0, 1]$. There are two reasons to do this. First, it is easier to visualize dimension and capacity, and second, there exist several techniques of smooth ergodic theory that can be applied in a very general context and are very helpful in understanding the problem.

In Section 2 we will consider the map $f(z)$ defined from $[0, 1]$ in itself such that for

$$z \in [0, 1/2], \qquad f(z) = 2z$$

$$z \in [1/2, 1], \qquad f(z) = 2(z - 1/2)$$

Consider the following partition of $[0, 1]$: $A = [0, 1/2)$ and $B = [1/2, 1]$. For each point $x \in [0, 1]$, associate the sequence x_n of zeros and ones, $(x_n)_{n \in \mathbb{N}}$, where

$$x_n = 1 \qquad \text{if and only if } f^n(x) \in A$$

$$x_n = 0 \qquad \text{if and only if } f^n(x) \in B$$

It is easy to see that for each $x \in [0, 1]$, the sequence (x_n) is the binary expansion of x. Consider the map such that for each $x \in [0, 1]$ associate $(x_n) = \tilde{g}(x) \in \{0, 1\}^{\mathbb{N}}$. This $\tilde{g}$ is a change of coordinates from the interval $[0, 1]$ to the set $\{0, 1\}^{\mathbb{N}}$ previously mentioned.

The shift map σ: $\{0, 1\}^{\mathbb{N}} \mapsto \{0, 1\}^{\mathbb{N}}$ such that for $\{u_0, u_1, u_2,...\}$ gives us as image the sequence $\{u_1, u_2, u_3,...\}$, clearly satisfying the equation

$$\sigma(\tilde{g}(x)) = \sigma \circ \tilde{g}(x) = \tilde{g} \circ f(x) = \tilde{g}(f(x))$$

Therefore, instead of analyzing the Bernoulli system σ in $\{0, 1\}^{\mathbb{N}}$, we can alternatively look for the system given by the map f acting on $[0, 1]$.

We can also transfer probabilities from the setting of $\{0, 1\}^{\mathbb{N}}$ to the setting of $[0, 1]$. Note that the arrangement $\{1, 1, 1, 1,...\} \in \{0, 1\}^{\mathbb{N}}$ is associated with the point $0 \in [0, 1]$. This is the different, but equivalent, setting we mentioned before.

The results presented in ref. 13 are reasonably general and some ideas can be applied in the Fisher model, as will be done in Section 2.

The results in ref. 13 are for rational maps. The map $f(z)$ considered here in fact can be seen as the complex polynomial z^2 acting on the unit circle. Any complex polynomial is a rational map, and in the particular case of the map z^2 the Julia set is the unit circle. Therefore, we are dealing here with a specific example of the general case considered in ref. 13.

In fact, Hofbauer considered essentially this simple transformation f and showed that if one considers a certain complicated potential, then, in some cases, the equilibrium state is not unique.[8] We will explore this result here, showing that in fact we have a phase transition.

We refer the reader to refs. 1, 2, 5, 6–17, 19, 21, and 22 for results about phase transition, equilibrium measures and some applications.

2. THE PROOF OF THE MAIN THEOREM

Consider $f(z)$ defined from $[0, 1]$ in itself as

$$z \in [0, 1/2], \qquad f(z) = 2z$$
$$z \in [1/2, 1], \qquad f(z) = 2(z - 1/2)$$

Note that the Liapunov number is constant equal $\log 2$, and $f(0) = 0$. Denote $M_0 = [1/2, 1]$, $M_1 = [1/4, 1/2]$, $M_2 = [1/8, 1/4]$, $M_3 = [1/16, 1/8]$,..., and $M_\infty = \{0\}$.

Consider also a_n a sequence of negative numbers such that $\lim_{n \to \infty} a_n = 0$ and $s_k = a_0 + a_1 + \cdots + a_k$, $k \in \mathbb{N}$, $n \in \mathbb{N}$.

Define g: $[0, 1] \to \mathbb{R}$ a scalar function such that $g(z) = a_k$ for $z \in M_k$ and $g(0) = 0$.

Note that the results of thermodynamic formalism in ref. 20 are for potentials g of a different kind than the ones we will consider here. For the potentials considered in ref. 20, the pressure is differentiable and equilibrium states are unique (no phase transition).

It follows from ref. 8 that under the hypothesis

$$\sum_{k=0}^{\infty} e^{S_k} > 1$$

there exists a unique equilibrium state for the variational problem associated with the pressure

$$P(g) = \sup_{\hat{v} \in M(f)} \left\{ h(\hat{v}) + \int g(z)\, d\hat{v}(z) \right\}$$

where $M(f)$ is the set of invariant probabilities for f and $h(\hat{v})$ is the entropy of $\hat{v} \in M(f)$ (see refs. 18, 20, and 23) for definitions).

The entropy plays the role of kinetic energy and the function g plays the role of an external potential term. In our case we can think as $g(z)$ associated with the temperature. Equilibrium measures (also called maximal pressure measures) play the role of Gibbs states.[19,20]

We can always transfer results from one setting to the other by means of the function $\tilde{g}$ defined in Section 1.

We consider g fixed and add a new parameter $B \in \mathbb{R}$, obtaining in this way a new scalar function Bg.

We assume here as in ref. 8 that

$$\sum_{k=0}^{\infty} (k+1) e^{S_k} < \infty$$

Remark. The case $\sum_{k=0}^{\infty} (k+1) e^{S_k} = \infty$ can be also analyzed, but should be seen as a second-order transition. We will not consider this case here (see ref. 17).

Now consider the parameter B. When we increase this value B, we find a first value B_0 such that

$$\sum_{k=0}^{\infty} e^{BS_k} = 1$$

In ref. 8 it is shown that for such B_0

$$0 = P(B_0 g) = \sup_{\hat{v} \in M(f)} \left[h(\hat{v}) + B_0 \int g(z) \, d\hat{v}(z) \right]$$

and there exist two equilibrium measures v and δ_0 (the delta Dirac with mass one in the point zero). Note that as $f^{-1}(0) = 0$, then δ_0 is in $M(f)$.

The Jacobian of the measure v is by definition the function defined for $z \in [0, 1]$, v-almost everywhere, such that

$$J(z) = \lim_{r \to 0} \frac{v(f(B(z, r)))}{v(B(z, r))}$$

It is well known that $h(v) = \int \log J(z) \, dv(z)$.

The notion of the Jacobian of a measure is quite natural. In the case where v (this does not happen here) is the Lebesgue measure λ, then the Jacobian would be the derivative of f.

We refer the reader to ref. 8 for the above facts.

It follows from the usual arguments in the thermodynamic formalism that there exists a positive function h such that

$$J(z)^{-1}=\frac{e^{B_0 g(z)}h(z)}{h(f(z))}$$

The function h is explicitly obtained in ref. 8. Therefore,

$$-\log J(z)=B_0\, g(z)+\log h(z)-\log h(f(z))$$

We can suppose that h is v-integrable in our model (see Section 3).

Consider now an external parameter $t\in\mathbb{R}$ and the one-parameter family of scalar functions

$$-t\log J(z)=tB_0\, g(z)+[+t\log h(z)]-t\log h(f(z))$$

As $M(f)$ is the set of invariant measures, then for any $\hat{v}\in M(f)$,

$$\int \log h(z)\, d\hat{v}(z)-\int \log h(f(z))\, d\hat{v}(z)=0$$

Therefore, for any $t\in\mathbb{R}$

$$P(tB_0\, g(z))=P(-t\log J(z))=\sup_{\hat{v}\in M(f)}\left[h(\hat{v})-t\int \log J(z)\, d\hat{v}(z)\right]$$

Denote $p(t)=P(tB_0\, g(z))$. Therefore, $p(1)=P(B_0\, g)=0$, and for $t<1$, $p(t)>0$. Note also that $p(0)=\log 2$ (see ref. 8).

From the theorem on p. 239 in ref. 8, for each $t<1$, there exists a unique equilibrium state v_t such that

$$h(v_t)-t\int \log J(z)\, dv_t(z)=h(v_t)+\int tB_0\, g(z)\, dv_t(z)=P(tB_0\, g)>0$$

We have also that for $t>1$, $P(tB_0\, g)=0$ and δ_0 is the only equilibrium state for $P(tB_0)$.

Given a measure $\hat{v}\in M(f)$, by definition, $\mathrm{HD}(\hat{v})$, the Hausdorff dimension of the measure $\hat{v}$, is the value[10]

$$\mathrm{HD}(\hat{v})=\inf\{\mathrm{HD}(A):\hat{v}(A)=1\}$$

Denote $h(t)=h(v_t)$, $\mathrm{HD}(t)=\mathrm{HD}(v_t)$, and $L(t)=t\int \log J(z)\, dv_t(z)$.

In the same way as in refs. 12 and 13, we have

$$p'(t)=-\int \log J(z)\, dv_t(z)$$

Note also that v_0 is the maximal measure, which in this case is the Lebesgue probability on [0, 1].

From the theorem of p. 236 of ref. 8, the measure $v_1 = v$ is not the Lebesgue probability λ. Therefore $h(v) < \log 2$.

The graph of $p(t)$ is shown in Fig. 1.

Remember that

$$h(v_1) = \int \log J(z)\, dv_1(z) = \int \log J(z)\, d\hat{v}(z)$$

Note that $P(t)$ is not linear for $t \in [0, 1]$.

Therefore, in the same way as in ref. 13, we are following a unique equilibrium v_t state until we find the critical value $t = 1$, where there appears another equilibrium state δ_0, which will be followed as a unique equilibrium state for $t > 1$.

Remark. In the case $\sum_{k=0}^{\infty} (k+1) e^{S_k} = \infty$, the derivative of p at 1 is zero and therefore we have a second-order transition. The reason is that the entropy of a δ-Dirac measure is zero.

We would like now to explain the analogy of the situation presented here with the case covered in ref. 13.

The Lebesgue measure λ (the maximal measure for f) plays the role of μ in ref. 13.

As we are considering here $P(-t \log J(z))$ instead of $P(-t \log |f'(z)|)$, we will have to introduce a scaling reference measure to measure balls of "size" ξ. We mean by this that for each $z \in [0, 1]$, a ball of "size" ξ with respect to v is in fact a ball of radius $\tilde{\xi}$ such that $v(B(z, \tilde{\xi})) = \xi$. This point was not noticed in ref. 13, because $|f'(z)|$ is like the Jacobian of Lebesgue measure.

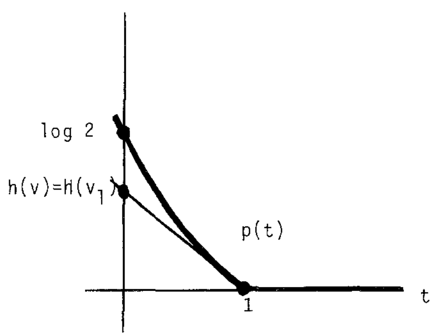

Fig. 1. Graph of the pressure $p(t)$.

Now that we have already presented the essential features of the model; we will proceed formally in an analogous way as in ref. 13.

The relation with the generalized dimension of the maximal entropy measure and Legendre–Fenchel transforms is the following: Given $\alpha \in \mathbb{R}$, find $t<1$ such that $p'(t) = -\alpha \log 2$. The value q is given by $q = \mathrm{HD}(t) - \alpha t$.

We suppose these three variables are related.

Now for each $q \in \mathbb{R}$, define

$$\mathscr{T}(q) = \frac{q - \mathrm{HD}(t)}{\alpha} \tag{$*$}$$

Now we will show how the value $\mathscr{T}(q)$ can be obtained as the analog of the generalized dimension as defined in ref. 13.

Suppose $q \in \mathbb{R}$ is given (and therefore t and α).

For each q fixed, our system recognizes just the measure v_t (and also the measure v) for t related to the value q by the relation above. For each $\xi > 0$, $\delta > 0$, consider A contained in the support of the measure v_t such that $v_t(A) > 1 - \delta$. Now consider for each $z \in A$ the value $\tilde{\xi}$ such that $v(B(z, \tilde{\xi})) = \xi$.

We point out here the essential difference from ref. 13. In ref. 13 we used $-\log |f'(z)|$ instead of $-\log J(z)$ because there we were looking for balls with radius (size) ξ, and here we are looking for balls with radius $\tilde{\xi}$ that have v-measure (size) ξ. This is the reason for the consideration of $v(B(z, \tilde{\xi})) = \xi$ and the introduction of a scaling reference measure. Now, from refs. 12, 13, 17, and 20 we have

$$-\int \log J(z)\, dv_t(z) = p'(t) = -\alpha \log 2$$

We have also from ref. 18, for z v_t-almost everywhere

$$\begin{aligned} -\log 2\alpha &= \lim_{n \mapsto \infty} \frac{1}{n} \log \prod_{j=0}^{n-1} [J(f^j(z))]^{-1} \\ &= \lim_{n \mapsto \infty} \frac{1}{n} \log v\left(B\left(z, \frac{1}{2^n}\right)\right) \end{aligned}$$

From this it follows from ref. 17 that $\exists A_1(\xi)$, $A_2(\xi)$ such that for z v_t-almost everywhere and $z \in A$, we have

$$A_1(\xi)\,\xi^\alpha < v(B(z, \xi)) < A_2(\xi)\,\xi^\alpha$$

where

$$\lim_{\xi \to 0} \frac{\log A_1(\xi)}{\log \xi} = 0$$

and

$$\lim_{\xi \to 0} \frac{\log A_2(\xi)}{\log \xi} = 0$$

We will explain the above claim in more detail.

We point out the following very important point: the measure v is not a homogeneous measure. Nevertheless, if we avoid points in right neighborhoods of 0 and 0.5, we have h bounded above and below. Using the reasoning of remark 5 in ref. 17, we have

$$v\left(B\left(z, \frac{1}{2^n}\right)\right) \sim n^{\Delta} \prod_{j=0}^{n-1} J(f^j(z))^{-1}$$

for some $\Delta \in \mathbb{R}$. Therefore, if we take A a large set and not covering right neighborhoods of 0 and 0.5 and supposing ξ of the form $1/2^n$, we obtain $A_1(\xi)$ and $A_2(\xi)$ as above.

The above argument can be done in a mathematically rigorous way, but we omit the details here. See ref. 17 and Section 3 for mathematical details.

Now, we point out that to cover A with balls of radius $\tilde{\xi}$ [such that $v(B(z, \tilde{\xi})) = \xi$] means to cover A with balls of radius $\xi^{(1/\alpha)}$. In fact, A_1 and A_2 will play no role here. Now we consider the maximal measure (Lebesgue probability λ in our case) of the ball $B(z, \tilde{\xi})$, that is, the value $\tilde{\xi} = \xi^{(1/\alpha)}$, because $\lambda(B(z, \tilde{\xi})) = \tilde{\xi}$.

Consider now the minimum number of balls necessary to cover A, denoted by $\bigcup_i B(z_i, \tilde{\xi}_i)$, where i has a range in a finite set (dependent on A, ξ, δ); consider the sum

$$\frac{\log \sum \lambda(B(z, \tilde{\xi}_i))^q}{\log \xi} = \frac{\log \sum (\tilde{\xi}_i)^q}{\log \xi} = \frac{\log \sum \xi^{(q/\alpha)}}{\log \xi}$$

Now, as in ref. 13, we consider all possible A such that $v_t(A) > 1 - \delta$ and consider the supremum of all possible sums. Finally, considering lim sup when $\delta \mapsto 1$ and $\xi \mapsto 0$, the number of such balls will increase exponentially as $\tilde{\xi}^{-\mathrm{HD}(t)} = \xi^{-\mathrm{HD}(t)/\alpha}$.[24] We refer the reader of ref. 24 and also to the final remarks of this paper for the rigorous mathematical setting.

Finally, we have from the same considerations of Theorem 4.4 in ref. 24 that the value of the above quotient goes to

$$\frac{-\mathrm{HD}(t)}{\alpha} + \frac{q}{\alpha} = \frac{q - \mathrm{HD}(t)}{\alpha}$$

(see also Proposition 1 in ref. 13).

This value agrees with $\mathscr{T}(q)$ as defined in $(*)$.

In the same way as in ref. 13, we have for $V(t)=P(-t)/\log 2$ the equation $V(\mathscr{T}(q))=q$, because

$$V(\mathscr{T}(q))=\frac{h(-\mathscr{T}(q))}{\log 2}+\frac{\mathscr{T}(q)}{\log 2}\int \log J(z)\, dv_{\mathscr{T}(q)}(z)$$

$$=\mathrm{HD}(-\mathscr{T}(q))-\frac{q-\mathrm{HD}(-\mathscr{T}(q))}{\alpha \log 2}\,p'(\mathscr{T}(q))=q$$

Therefore, as $p(t)$ has lack of differentiability on $t=1$, the same property holds for $\mathscr{T}(q)$, because

$$p(-\mathscr{T}(q))=\log 2q$$

The probabilities v_t should be seen as the law of choosing the centers of the balls to cover the set $[0, 1]$. The probability v is the one where the transition to δ_0 in the critical parameter $t=1$ occurs.

In the terminology of ref. 13 that is borrowed from large-deviation theory, we have a transition at level 2, that is, at the level of measures. ∎

Now we would like to make some final remarks.

The concepts of F. Ledrapier and L. S. Young of capacity and Hausdorff dimension of a measure allow some more flexibility in choosing at random the center of the balls in the definition of generalized dimension. The probability laws are given by the v_t measures.

The setting of generalized dimension (observables) is more natural for analyzing problems of phase transitions than the setting of pressure. Anyway, there is a strong relationship between both concepts: $P(-\mathscr{T}(q))=\log 2q$.

The relationship $p'(t)=-\log 2\alpha$ is a kind of Legendre–Fenchel transform relationship of large-deviation theory.[14,16]

Note that the sudden appearance of a delta-Dirac with mass one in the point $0\in[0, 1]$ can be seen, as mentioned in Section 1, as a sudden magnetization in the critical parameter. Remember that $0\in[0, 1]$ is associated with the arrangement $\{1, 1, 1, 1,...\}$.

The generalized dimension is related to capacity and the results obtained in the setting of the pressure and thermodynamic formalism are in general related to the Hausdorff dimension.

It is essential to have some connection between Hausdorff dimension and capacity. We postponed this consideration to the end of the paper in order to describe the model in a more transparent way. Now we will provide the reader with a more precise and accurate formalization of the concepts with which we are dealing.

It is well known that the values of capacity and dimension are not always the same for general sets. As we have already said, the concept of generalized dimension requires the use of capacity in one way or another in order to have the meaning that corresponds to partitions and thermodynamic limits.

F. Ledrapier introduced the concept of capacity for measures, and the result of L. S. Young relates in a very general situation the concept of Hausdorff dimension and capacity for a measure. This is an essential element for the understanding of a phase transition.

We borrow the following statement from ref. 24: If the dimension of a set is to be related to the μ-entropy and μ-exponents of a measure μ invariant for f, that notion of dimension must be sensitive to the "good points" of μ. Capacity, however, does not distinguish between a set and its closure. F. Ledrapier made the following modification to correct this insensitivity. Let μ be a Borel probability measure on X. Then define

$$\underline{C}(\mu)=\sup_{\delta\to 0}\inf_{\substack{Y\subset X\\ \mu(Y)\geqslant 1-\delta}}\underline{C}(Y)$$

and

$$\bar{C}(\mu)=\sup_{\delta\to 0}\inf_{\substack{Y\subset X\\ \mu(Y)\geqslant 1-\delta}}\bar{C}(Y)$$

where

$$\underline{C}(Y)=\liminf_{\xi\to 0}\frac{\log N(\xi)}{\log(1/\xi)}$$

$$\bar{C}(Y)=\limsup_{\xi\to 0}\frac{\log N(\xi)}{\log(1/\xi)}$$

and $N(\xi)$ is the minimum number of ξ-balls needed to cover Y.

The result of L. S. Young claims that if for x μ-almost everywhere

$$\lim_{p\to 0}\frac{\log\mu(B(x,p))}{\log p}=\alpha_0$$

then $\bar{C}(\mu)=\underline{C}(\mu)=\alpha_0=\mathrm{HD}(\mu)$.

From ref. 18, the hypothesis assumed in the above result is true; therefore, we can use the conclusion of L. S. Young's result, and the reader can see how this point was essential in the correct establishment of the concept of generalized dimension of the maximal measure λ.

We now give a trivial example to elucidate some of the features of the above definition.

Suppose X is the sequence of real numbers of the form

$$X = \{1, 2^{-1}, 3^{-1}, 4^{-1}, \dots, n^{-1}, \dots\} \cup \{0\}$$

Assume μ is a probability such that the mass $\mu(\{n^{-1}\}) = 2^{-n}$ and $\mu(\{0\}) = 0$.

The Hausdorff dimension of μ is trivially zero because X has dimension zero.

Note also that for x μ-almost everywhere

$$\lim_{p \to 0} \frac{\log \mu(B(x, p))}{\log p} = 0$$

The capacity dimension of X is one, but we claim that the value

$$\bar{C}(\mu) = \underline{C}(\mu) = 0$$

The claim follows easily from the fact that, given $\delta > 0$, as $\sum_{n=0}^{\infty} 2^{-n}$ converges, then there exists $N > 0$ such that $\sum_{n=N+1}^{\infty} 2^{-n} < \delta$. Therefore, the set $Y = \{1, 2^{-1}, \dots, N^{-1}\}$ satisfies $\mu(Y) > 1 - \delta$, but has capacity zero.

Even if the example is not directly related to our main result, we believe it gives an idea of the reason for using the above considerations. We can get rid of the "bad" points.

Generally speaking, the procedure physicists use most of the time to analyze equilibrium states in the lattice $\mathbb{N}$ is the following. They consider for each fixed $n \in \mathbb{N}$ the finite lattice $\{0, 1, 2, \dots, n-1, n\}$, solve the problem of finding an equilibrium state in the finite lattice, and then take the limit as n goes to infinity, obtaining in the limit the equilibrium state. One of the differences in the model of the thermodynamic formalism is that one works with measures on $\{0, 1\}^{\mathbb{N}}$ and this means that we are not truncating the lattice $\mathbb{N}$ in a finite lattice and then taking the limit. Therefore, in thermodynamic formalism models, one has to use the notion of the entropy of a measure invariant for the shift, which is more sophisticated than the analogous one for the finite lattice. Some essential elements presented in the model here in the setting of thermodynamic formalism have no meaning or are difficult to translate for the other procedure of truncating the lattice $\mathbb{N}$ in finite pieces and taking thermodynamic limits. Note that the classic references for thermodynamic formalism do not consider phase transition problems (see refs. 20 and 23).

In the example presented here, the Lebesgue measure on the interval (the maximal entropy measure for f) plays the role of the Boltzmann factor.

3. AN EXAMPLE

We present a class of examples where explicit computations can be obtained. We consider a slight generalization of a potential g (we use the notation of Section 2) considered by Hofbauer.[8]

For each value of γ we will have an example where the above considerations apply.

Consider $3<\gamma$ and

$$a_k = -\gamma \log\left(\frac{k+2}{k+1}\right) \quad \text{for} \quad k \geqslant 1$$

and

$$a_0 = -\log\left(1 + \sum_{k=1}^{\infty} e^{a_1 + \cdots + a_k}\right) \tag{**}$$

An easy computation shows that

$$\sum_{k=0}^{\infty} e^{s_k} = 1 \quad \text{and} \quad \sum_{1=k}^{\infty} (k+1) l^{S_k} < \infty$$

Therefore, given $3<\gamma$, the value $B_0 = B_0(\gamma)$ is equal to 1 (see the notation of Section 2).

Following Hofbauer,[8] we have that the function h (see Section 2) is an eigenfunction of the Ruelle–Perron–Frobenius operator and is given in the following way: Define first

$$r_k = e^{a_0}\left(\frac{k+2}{2}\right)^{-\gamma}$$

for $1 \leqslant k$ and $r_0 = e^{a_0}$. Consider now $u = (\sum_{k=1}^{\infty} k r_{k-1})^{-1}$ and finally from ref. 8 we have that

$$h(z) = u r_k^{-1} \sum_{i=k}^{\infty} r_k \quad \text{for} \quad z \in M_k$$

As $3<\gamma$, h can be shown to be v-integrable.

The Jacobian of the measure v is therefore given by the formula

$$J(z)^{-1} = \left(\frac{k+2}{k+1}\right)^{-\gamma} \frac{r_k^{-1}}{r_{k-1}^{-1}} \frac{\sum_{i=k}^{\infty} r_i}{\sum_{i=k-1}^{\infty} r_i} \quad \text{for} \quad z \in M_k,\ k>1$$

The v measure of the set M_k is given by

$$v(M_k) = u \sum_{i=k}^{\infty} r_i$$

Finally, the entropy of v is obtained in the following way.

Consider the sets Q_k, $k \geqslant 1$, such that $Q_k \cap M_k = \varnothing$ and $f(Q_k) = f(M_k) = M_{k-1}$. Note that

$$I = \left(\bigcup_{k=1}^{\infty} M_k \right) \cup \left(\bigcup_{k=1}^{\infty} Q_k \right)$$

Note also from the invariance of μ that $v(M_k) + v(Q_k) = v(M_{k-1})$; we have

$$v(Q_k) = ur_{k-1}$$

In each Q_k the value of $J(z)$ is equal to $[1 - J(x)^{-1}]^{-1}$, where $x \in M_k$.

From the above, we can compute the entropy

$$h(v) = \int \log J(z)\, dv(z)$$

$$= \sum_{\substack{k=1 \\ z_k \in M_k}}^{\infty} \log J(z_k)\, v(M_k) + \sum_{\substack{k=1 \\ x_k \in Q_k}}^{\infty} \log J(x_k)\, v(Q_k)$$

The same kind of computation also works for v_t, that is, we are able to compute $h(v_t)$, $t \leqslant 1$. In order to do that, we first compute the Jacobian of v_t from the considerations of page 226 and 228 in ref. 8.

Note the important fact that for a fixed valued of γ in the variational problem

$$P\{-t \log J(z)\} = P\{tg(z)\} = \sup_{\hat{v} \in M(f)} \left\{ h(\hat{v}) - t \int \log J(z)\, d\hat{v}(z) \right\}$$

we are considering potentials $tg(z)$ that are of the form of Hofbauer

$$tg(z) = \gamma t \log \left(\frac{k+2}{k+1} \right) \qquad \text{for} \quad z \in M_k,\ k > 1$$

The only difference is that for such $tg(z)$ we have $\sum_{k=0}^{\infty} e^{S_k} < 1$ [remember that ta_0 is no longer in the form (**)]; therefore, the same explicit computations presented in ref. 8 apply.

From these considerations we can say that $p'(t) = -\int \log J(z)\, dv_t(z)$ can be explicitly computed and therefore, given $\alpha \in \mathbb{R}$, we can in principle obtain t and finally v_t. This measure v_t can be obtained explicitly, as mentioned before (see page 226 and 228 of ref. 8).

Finally, we can also compute the generalized dimension directly because we know explicitly $v_t(M_k)$ for $k \in \mathbb{N}$. In this way (following the notation of Section 2), given $\xi > 0$ and $\delta > 0$, consider A of the form $A = \bigcup_{k \in \Phi} M_k$, where Φ is a set of indices, $\Phi \subset \mathbb{N}$.[17]

As we know, $v_t(M_k)$ for $k \in \Phi$; then we know when $v_t(A) > 1 - \delta$ and therefore all computations can be done.[17]

In conclusion, from the above considerations we can say that in the above class of examples it is possible to compute with precision all values involved. The computations are, of course, not simple, but this is the nature of the problem.

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NOTE ADDED IN PROOF

In ref. 17 the following result related with Section 3 of this paper is obtained: consider $\gamma > 1$ fixed, then the pressure $p(t)$ satisfies a functional equation of the kind

$$\zeta(\gamma)^t = \sum_{n=1}^{\infty} \frac{e^{-np(t)}}{n^{t\gamma}}$$

where $\zeta(\gamma)$ is the Riemann zeta function. As far as we know this functional equation was not noticed before. From the above equation we give a rigorous mathematical proof that

$$p(t) \sim h(v)(1-t) + (1-t)^{\gamma-1} + A(1-t)^2$$

when $t \sim 1$ and $3 < \gamma < 4$.

The value $\gamma - 1$ is the critical exponent of transition, also considered in the physics literature by M. Fisher, B. Felderhof and X.-J. Wang.

We will also prove in ref. 17 rigorous mathematical results for the pressure of the Manneville–Pomeau map.

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